

The regular state in higher order gravity

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Abstract

We start with a formal series expansion of the regular asymptotic solution of the higher order gravity theory derived from the quadratic lagrangian $R + \epsilon R^2$ in vacuum, and build a time development as a solution of an initial value formulation of the theory. We show that the resulting solution, if analytic, contains the right number of free functions to qualify as a general solution of the theory. We further show that any regular analytic solution which satisfies the constraints and evolution equations can be given in the form of such an asymptotic formal series expansion.

1 Introduction

Ever since the fundamental realization that Einstein's equations can be equivalently studied as a geometric, nonlinear system of evolution equations with constraints, there has been a considerable stream of problems in general relativity of a classical dynamical nature crucially depending on the notion of evolution from prescribed initial data (cf. [1] and references therein). In particular, for the so-called cosmological case, at least since the investigations of Lifshitz and Khalatnikov in relativistic cosmology [2], perturbative approaches to the question of genericity of cosmological solutions constructed asymptotically from given initial data, have been advanced in various directions in general cosmological theory. This was the way to prove the existence of vacuum, regular

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cosmological solutions in general relativity having the required number to qualify as general ones, and also the non existence of a general singular, radiation solution in the same context, cf. [2, 3], eventually leading to the realization that the initial state was of a more complex nature [4, 5, 6]. These first results were based on an approximation technique that consisted of several steps such as writing down a suitable expansion of the metric, substituting it to the field equations and counting the number of free (or arbitrary) functions needed to make the whole scheme consistent.

The original perturbative approximation scheme mentioned above proved to be especially fertile, and in fact it is being exploited in various directions ever since. It was used by Starobinski to study solutions of the Einstein equations with a positive cosmological constant, especially with regard to the question of the asymptotic stability of de Sitter space used in inflationary scenarios [7, 8]. The formal series expansions were used more recently in [9] to study the genericity question in relativistic cosmologies with a general equation of state $p = w\rho$, where it was proved that in order to be able to construct a general singular solution initially, certain restrictions on the fluid parameter w are needed. The ultrastiff case was recently considered in a more rigorous way using Fuchsian techniques in Ref. [10]. Formal series were also used in [11] to study the more involved problem of perturbing an FRW universe containing two suitable fluids, and in [12] where it was shown how to construct a sudden singularity with the genericity properties of a general solution, a situation met previously only in the ‘no-hair’ behaviour of inflation. Extensions to certain higher order gravity theories have also been considered in a formal series context, especially in connection with the stability of de Sitter space under generic perturbations in these theories, cf. [13, 14].

It is interesting that the original approximation scheme of formal perturbation series has been further refined and applied in various situations in cosmology, cf. [15, 16, 17] and references therein. More recently, Rendall has put the original formal series expansion techniques used by Starobinski [7] for the stability of de Sitter space, in a more rigorous basis and was able to prove various interesting theorems concerning function counting and formal series solutions of the Einstein equations with a positive cosmological constant in an ‘initial value problem’ spirit, cf. [18] (see also [19]).

In this paper, we are interested in the possible genericity of *regular* solutions defined by formal series expansions in the context of vacuum models in higher order gravity derived from the analytic lagrangian $R + \epsilon R^2$ theory. In particular, we prove that the higher order gravity field equations in vacuum admit a unique solution in the form of a regular formal power series expansion which contains the right number of free functions to qualify as a general solution of the system. This requires a careful function counting technique, and for this purpose it is necessary to develop a formulation of the theory as a system of evolution equations with constraints.

The plan of this paper is as follows. In the next Section, we write the higher order gravity field equations as a system of first order (ADM-type) evolution equations and constraints, as in an initial value formulation of the theory, and show that this system has the Cauchy-Kovalevski property. This allows us to count the true degrees of freedom necessary for any analytic solution to be a general one. In Section 3, we introduce a regular formal series representation of the metric and prove our main result about the existence and uniqueness of a regular generic perturbation in higher order gravity starting from given formal asymptotic data. Finally in the last Section, we present our conclusions and further discussion of these results. In Appendix A, we give more details about certain crucial steps in the proof of the Cauchy-Kovalevski property of our main system, while in Appendix B we present the full expressions of certain curvature components in terms of asymptotic data.

2 Function counting

In this Section, we are interested in counting the true degrees of freedom of the $R + \epsilon R^2$ theory in vacuum. This requires a splitting formulation of the theory into evolution equations and constraints, which in turn relies on certain technical details that the resulting dynamical system has to satisfy for the whole scheme to be consistent. The main technical result is given at the end of this Section, Theorem 2.1. We then count the degrees of freedom of the theory. In the Appendix A, we give more details of the proof of the main theorem of this Section.

We consider a spacetime (\mathcal{V}, g) where $\mathcal{V} = \mathbb{R} \times \mathcal{M}$, with \mathcal{M} being an orientable 3-manifold, the submanifolds $\mathcal{M}_t = \{t\} \times \mathcal{M}$, $t \in \mathbb{R}$, are spacelike and g is a Lorentzian metric, analytic and with signature $(+, -, -, -)$ ¹. We take a *Cauchy adapted frame* $e_i = (e_0, e_\alpha)$ with e_α tangent to the space slice \mathcal{M}_t and e_0 orthogonal to it. The *dual coframe* $\theta^i = (\theta^0 = dt, \theta^\alpha = dx^\alpha + \beta^\alpha dt)$, where the tangent vector β^α is the usual *shift*, leads to the standard general form of the metric g ,

$$ds^2 = N^2 dt^2 - \bar{g}_{\alpha\beta}(t) (dx^\alpha + \beta^\alpha dt) (dx^\beta + \beta^\beta dt). \quad (2.1)$$

Here N is a positive function, the *lapse*, and we assume that all metrics $\bar{g}_{\alpha\beta}(t)$ are complete Riemannian metrics. We could continue using a bar to denote spatial tensors², in a Cauchy adapted frame we would have $\bar{g}_{\alpha\beta} = -g_{\alpha\beta}$ and $\bar{g}^{\alpha\beta} = -g^{\alpha\beta}$. However, we more frequently write $\gamma_{\alpha\beta} = -g_{\alpha\beta}$. Below we shall deal exclusively with the simplest gauge choice, $N = 1, \beta = 0$, which means that $g_{00} = 1, g_{0\alpha} = 0$, and local coordinates are *adapted* to the product structure on \mathcal{M}_t , $(x^i = (t, x^\alpha))$ (in this case, one sometimes speaks of a *synchronous* system of local coordinates).

¹Our conventions are those of [3].

²We generally follow closely the terminology of the fundamental treatise [1].

Our starting point is the vacuum field equations of the higher order gravity theory derived from an analytic lagrangian $f(R)$, that is the equations

$$L_{ij} = f'(R)R_{ij} - \frac{1}{2}f(R)g_{ij} - \nabla_i \nabla_j f'(R) + g_{ij} \square_g f'(R) = 0. \quad (2.2)$$

We shall be concerned below with the quadratic theory $f(R) = R + \epsilon R^2$, for which, the field equations (2.2) in a Cauchy adapted frame split as follows:

$$L_{00} = (1 + 2\epsilon R)R_{00} - \frac{1}{2}(1 + \epsilon R)R + 2\epsilon g^{\alpha\beta} \nabla_\alpha \nabla_\beta R = 0, \quad (2.3)$$

$$L_{0\alpha} = (1 + 2\epsilon R)R_{0\alpha} - 2\epsilon \nabla_0 \nabla_\alpha R = 0, \quad (2.4)$$

$$L_{\alpha\beta} = (1 + 2\epsilon R)R_{\alpha\beta} - \frac{1}{2}(1 + \epsilon R)Rg_{\alpha\beta} - 2\epsilon \nabla_\alpha \nabla_\beta R + 2\epsilon g_{\alpha\beta} \square_g R = 0. \quad (2.5)$$

The components of the Ricci tensor are

$$R_{00} = -\frac{1}{2}\partial_t K - \frac{1}{4}K_\alpha^\beta K_\beta^\alpha, \quad (2.6)$$

$$R_{0\alpha} = \frac{1}{2}(\nabla_\beta K_\alpha^\beta - \nabla_\alpha K), \quad (2.7)$$

and also

$$R_{\alpha\beta} = P_{\alpha\beta} + \frac{1}{2}\partial_t K_{\alpha\beta} + \frac{1}{4}KK_{\alpha\beta} - \frac{1}{2}K_\alpha^\gamma K_{\beta\gamma}, \quad (2.8)$$

where $K = \text{tr}K_{\alpha\beta}$. Here, the extrinsic curvature is defined by the *first variational equation*

$$\partial_t \gamma_{\alpha\beta} = K_{\alpha\beta}, \quad (2.9)$$

and $P_{\alpha\beta}$ denotes the three-dimensional Ricci tensor associated with $\gamma_{\alpha\beta}$. Further, we define the *acceleration tensor* $D_{\alpha\beta}$ through the *second variational equation*

$$\partial_t K_{\alpha\beta} = D_{\alpha\beta}, \quad (2.10)$$

and we also introduce the *jerk tensor* (3rd order derivatives) $W_{\alpha\beta}$ through the *jerk equation*:

$$\partial_t D_{\alpha\beta} = W_{\alpha\beta}. \quad (2.11)$$

The space tensors $D_{\alpha\beta}$ and $W_{\alpha\beta}$ are obviously symmetric and because of the fourth order nature of the higher order field equations, they play an important role in what follows.

In terms of the acceleration tensor $D_{\alpha\beta}$, the Ricci tensor splittings given above become

$$R_{00} = -\frac{1}{2}D + \frac{1}{4}K_\alpha^\beta K_\beta^\alpha, \quad (2.12)$$

$$R_{0\alpha} = \frac{1}{2}(\nabla_\beta K_\alpha^\beta - \nabla_\alpha K), \quad (2.13)$$

$$R_{\alpha\beta} = P_{\alpha\beta} + \frac{1}{2}D_{\alpha\beta} + \frac{1}{4}KK_{\alpha\beta} - \frac{1}{2}K_{\alpha}^{\gamma}K_{\beta\gamma}, \quad (2.14)$$

while for the scalar curvature we obtain,

$$R = -P - D - \frac{1}{4}K^2 + \frac{3}{4}K_{\beta}^{\alpha}K_{\alpha}^{\beta}, \quad (2.15)$$

where $P = \text{tr}P_{\alpha\beta}$, $D = \text{tr}D_{\alpha\beta}$. If we substitute these forms into the identities (2.3),(2.4), we find the following equations, which, in obvious analogy with the situation in general relativity, we call *constraints*:

Hamiltonian Constraint

$$\begin{aligned} \mathcal{C}_0 = & \frac{1}{2}P + \frac{1}{8}K^2 - \frac{1}{8}K^{\alpha\beta}K_{\alpha\beta} + \\ & \epsilon[-\frac{1}{2}P^2 - \frac{1}{4}PK^2 + \frac{1}{4}PK^{\alpha\beta}K_{\alpha\beta} - \frac{1}{32}K^4 + \frac{1}{16}K^2K^{\alpha\beta}K_{\alpha\beta} + \\ & \frac{3}{32}(K^{\alpha\beta}K_{\alpha\beta})^2 - \frac{1}{2}DK^{\alpha\beta}K_{\alpha\beta} + \frac{1}{2}D^2 - \\ & 2\gamma^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}(-P - \frac{1}{4}K^2 + \frac{3}{4}K^{\gamma\delta}K_{\gamma\delta} - D)] = 0, \end{aligned} \quad (2.16)$$

Momentum Constraint

$$\begin{aligned} \mathcal{C}_{\alpha} = & \frac{1}{2}(\nabla_{\beta}K_{\alpha}^{\beta} - \nabla_{\alpha}K) + \\ & \epsilon[(-P - \frac{1}{4}K^2 + \frac{3}{4}K_{\gamma}^{\delta}K_{\delta}^{\gamma} - D)(\nabla_{\beta}K_{\alpha}^{\beta} - \nabla_{\alpha}K) - \\ & \nabla_{\alpha}(-2\partial_t P + KK^{\gamma\delta}K_{\gamma\delta} - 3K_{\gamma}^{\beta}K_{\delta}^{\gamma}K_{\beta}^{\delta} - KD + 5K^{\gamma\delta}D_{\gamma\delta} - 2W)] = 0 \end{aligned} \quad (2.17)$$

The constraints show that the *initial data* $(\gamma_{\alpha\beta}, K_{\alpha\beta}, D_{\alpha\beta}, W_{\alpha\beta})$ cannot be chosen arbitrarily and must satisfy the equations (2.16) and (2.17) on each slice \mathcal{M}_t ³.

Further, any *initial data set* $(\mathcal{M}_t, \gamma_{\alpha\beta}, K_{\alpha\beta}, D_{\alpha\beta}, W)$ must satisfy, apart from the equations (2.9) (velocity equation), (2.10) (acceleration equation), (2.11) (jerk equation), the following evolution

³It is to be noted that because of the structure of the last term in equation (2.5), the time derivative of the jerk tensor $W_{\alpha\beta}$ appears only for its trace, whereas the time derivatives of the other components of $W_{\alpha\beta}$ do not appear. This is reminiscent of a similar fact in the FRW equations for universe with scale factor a and filled with a perfect fluid in general relativity. There, there appears no time derivative of the pressure in the Friedman equations. Of course, one does not count the pressure as an initial value in the evolution which is determined by, say, a, \dot{a}, ρ as unknowns. Consequently, below we shall write $(\mathcal{M}_t, \gamma_{\alpha\beta}, K_{\alpha\beta}, D_{\alpha\beta}, W)$ for the initial data of the theory.

equation, called the *snap* equation:

$$\begin{aligned}
\partial_t W = & \frac{1}{6\epsilon} \left(\frac{1}{2} P + \frac{1}{8} K^2 - \frac{5}{8} K^{\alpha\beta} K_{\alpha\beta} + D \right) + \\
& \frac{1}{6} \left[P^2 + \frac{1}{4} P K^2 - \frac{1}{4} P K^{\alpha\beta} K_{\alpha\beta} + \frac{1}{32} K^4 - \frac{1}{16} K^2 K^{\alpha\beta} K_{\alpha\beta} - \right. \\
& 6 K K_{\beta}^{\alpha} K_{\gamma}^{\beta} K_{\alpha}^{\gamma} - \frac{99}{32} (K^{\alpha\beta} K_{\alpha\beta})^2 + 27 K_{\beta}^{\alpha} K_{\gamma}^{\beta} K_{\delta}^{\gamma} K_{\alpha}^{\delta} + 9 K K^{\alpha\beta} D_{\alpha\beta} - \\
& 57 K_{\beta}^{\alpha} K_{\gamma}^{\beta} D_{\alpha}^{\gamma} + \frac{13}{2} D K^{\alpha\beta} K_{\alpha\beta} - \frac{7}{2} D^2 + 15 D^{\alpha\beta} D_{\alpha\beta} - 3 K W + \\
& 15 K^{\alpha\beta} W_{\alpha\beta} - 6 \partial_t (\partial_t P) - \\
& \left. 4 \gamma^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} (-P - D + \frac{3}{4} K^{\alpha\beta} K_{\alpha\beta} - \frac{1}{4} K^2) \right] \tag{2.18}
\end{aligned}$$

The four evolution equations (2.9), (2.10), (2.11) and (2.18) are the higher order gravity analogues of the ADM equations of general relativity, and together with the constraints (2.16) and (2.17) describe the *time development* (\mathcal{V}, g) of any initial data set $(\mathcal{M}_t, \gamma_{\alpha\beta}, K_{\alpha\beta}, D_{\alpha\beta}, W)$ in higher order gravity theories. To be able to complete the function counting argument of this Section, we need to prove that these equations are well defined in the sense that there is a well defined Cauchy problem, at least for the analytic case.

To be concrete, one needs to prove that these equations are of the Cauchy-Kovalevski type, that is there are no time derivatives in the constraints and the derivatives of the unknowns are (through the evolution equations) analytic functions of the coordinates, the unknowns, and their first and second *space* derivatives. This is shown in the Appendix. From this result and the Cauchy-Kovalevski theorem (cf. [1], Appendix V), we are immediately led to the following result, local Cauchy problem (analytic case):

Theorem 2.1 *For $N = 1, \beta = 0$, if we prescribe analytic initial data $(\gamma_{\alpha\beta}, K_{\alpha\beta}, D_{\alpha\beta}, W)$ on some initial slice \mathcal{M}_0 , then there exists a neighborhood of \mathcal{M}_0 in $\mathbb{R} \times \mathcal{M}$ such that the evolution equations (2.9), (2.10), (2.11) and (2.18) have an analytic solution in this neighborhood consistent with these data. This analytic solution is the development of the prescribed initial data on \mathcal{M}_0 if and only if these initial data satisfy the constraints.*

The last part of this theorem follows most easily from the conformal equivalence theorem of higher order gravity theories [20], that is working in the Einstein frame representation, and a theorem on symmetric hyperbolic systems, cf. [1], pp. 150-1, and also [21]. Indeed, in the Einstein frame representation, the theory is general relativity plus a self-interacting scalar field, and the system becomes one of the form

$$R_0^0 = -\frac{1}{2} \partial_t K - \frac{1}{4} K_{\alpha}^{\beta} K_{\beta}^{\alpha} = 8\pi k (T_0^0 - \frac{1}{2} T), \tag{2.19}$$

$$R_\alpha^0 = \frac{1}{2}(\nabla_\beta K_\alpha^\beta - \nabla_\alpha K) = 8\pi k T_\alpha^0, \quad (2.20)$$

$$R_\alpha^\beta = -P_\alpha^\beta - \frac{1}{2\sqrt{\gamma}}\partial_t(\sqrt{\gamma}K_\alpha^\beta) = 8\pi k(T_\alpha^\beta - \frac{1}{2}\delta_\alpha^\beta T), \quad (2.21)$$

and the wave equation $\nabla^i \nabla_i \phi - V'(\phi) = 0$ for ϕ having the particular scalar field potential given in [20]. This system is of the form given in Thm. 4.1 of [1], p. 150, from which the result follows.

We are now ready to count the true degrees of freedom of the higher order gravity theories we consider. There are 19 relations, that is 18 by the three evolution equations (2.9), (2.10), (2.11) and one from (2.18). But we have the freedom to perform 4 diffeomorphism changes and we also have the 4 constraints (2.16) and (2.17). Hence, the vacuum theory has $19 - 4 - 4 = 11$ degrees of freedom. This in turn implies that any solution with 11 free functions has the same degree of generality with a general solution of the theory.

3 Genericity of regularity

In this Section we perform a perturbative analysis of our basic system of higher order equations, namely, the evolution equations (2.9), (2.10), (2.11) and (2.18) together with the constraints (2.16) and (2.17). In particular, we assume a regular formal series representation of the spatial metric of the form

$$\gamma_{\alpha\beta} = \gamma_{\alpha\beta}^{(0)} + \gamma_{\alpha\beta}^{(1)} t + \gamma_{\alpha\beta}^{(2)} t^2 + \gamma_{\alpha\beta}^{(3)} t^3 + \gamma_{\alpha\beta}^{(4)} t^4 + \dots \quad (3.1)$$

where the $\gamma_{\alpha\beta}^{(0)}, \gamma_{\alpha\beta}^{(1)}, \gamma_{\alpha\beta}^{(2)}, \gamma_{\alpha\beta}^{(3)}, \gamma_{\alpha\beta}^{(4)}, \dots$ are functions of the space coordinates. Because of the order of the higher order gravity equations, we shall be interested only in the part of the formal series shown, that is up to order four, and because of that we shall often drop the dots at the end of the various expressions to simplify the overall appearance⁴. Thus before substitution to the evolution and constraint higher order equations, the expression (3.1) contains 30 degrees of freedom. Note that setting $\gamma_{\alpha\beta}^{(0)} = \delta_{\alpha\beta}$ and $\gamma_{\alpha\beta}^{(n)} = 0, n > 0$, we have Minkowski space included here as an exact solution of the equations, and so our perturbation analysis covers also that case.

The problem we are faced with in this Section is what the initial number of thirty free functions becomes after the imposition of the higher order evolution and constraint equations, that is how it finally compares with the 11 degrees of freedom that any general solution must possess as shown in the previous Section. Put it more precisely, given data $a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta}, e_{\alpha\beta}$, analytic functions

⁴Differentiation of such formal series with respect to either space or the time variables is defined term by term, whereas multiplication of two such expressions results when the various terms are multiplied and terms of same powers of t are taken together.

of the space coordinates, such that the coefficients $\gamma_{\alpha\beta}^{(\mu)}$, $\mu = 0, \dots, 4$, are prescribed,

$$\gamma_{\alpha\beta}^{(0)} = a_{\alpha\beta}, \quad \gamma_{\alpha\beta}^{(1)} = b_{\alpha\beta}, \quad \gamma_{\alpha\beta}^{(2)} = c_{\alpha\beta}, \quad \gamma_{\alpha\beta}^{(3)} = d_{\alpha\beta}, \quad \gamma_{\alpha\beta}^{(4)} = e_{\alpha\beta}, \quad (3.2)$$

how many of these data are truly independent when (3.1) is taken to be a possible solution of the evolution equations (2.9), (2.10), (2.11) and (2.18) together with the constraints (2.16) and (2.17)?

To proceed, we shall need the formal expansion of the reciprocal tensor $\gamma^{\alpha\beta}$, where $\gamma_{\alpha\beta}\gamma^{\beta\gamma} = \delta_{\alpha}^{\gamma}$. Using this, the various coefficients $\gamma^{(\mu)\alpha\beta}$, $\mu = 0, \dots, 4$, of t in the expansion $\gamma^{\alpha\beta} = \sum_{n=0}^{\infty} \gamma^{(n)\alpha\beta} t^n$ are found to be:

$$\begin{aligned} \gamma^{\alpha\beta} = & a^{\alpha\beta} - b^{\alpha\beta}t + \left(b^{\alpha\gamma}b_{\gamma}^{\beta} - c^{\alpha\beta}\right)t^2 + \left(-d^{\alpha\beta} + b^{\alpha\gamma}c_{\gamma}^{\beta} - b^{\alpha\gamma}b_{\gamma}^{\delta}b_{\delta}^{\beta} + c^{\alpha\gamma}b_{\gamma}^{\beta}\right)t^3 \\ & + \left(-e^{\alpha\beta} + b^{\alpha\gamma}d_{\gamma}^{\beta} - b^{\alpha\gamma}b_{\gamma}^{\delta}c_{\delta}^{\beta} + c^{\alpha\gamma}c_{\gamma}^{\beta} + d^{\alpha\gamma}b_{\gamma}^{\beta} - b^{\alpha\gamma}c_{\gamma}^{\delta}b_{\delta}^{\beta} + b^{\alpha\gamma}b_{\gamma}^{\delta}b_{\delta}^{\epsilon}b_{\epsilon}^{\beta} - c^{\alpha\gamma}b_{\gamma}^{\delta}b_{\delta}^{\beta}\right)t^4. \end{aligned} \quad (3.3)$$

Note that $a_{\alpha\beta}a^{\beta\gamma} = \delta_{\alpha}^{\gamma}$ and the indices of $b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta}, e_{\alpha\beta}$ are raised by $a^{\alpha\beta}$. For any tensor X , using the formal expansion (3.1), we can recursively calculate the coefficients in the expansion

$$X_{\alpha\beta} = X_{\alpha\beta}^{(0)} + X_{\alpha\beta}^{(1)}t + X_{\alpha\beta}^{(2)}t^2 + X_{\alpha\beta}^{(3)}t^3 + X_{\alpha\beta}^{(4)}t^4 + \dots, \quad (3.4)$$

in particular we can write down a general iterated formula for the n -th order term, $X_{\alpha\beta}^{(n)}$. For instance, for the extrinsic curvature $K_{\alpha\beta}$, writing

$$K_{\alpha\beta} = K_{\alpha\beta}^{(0)} + K_{\alpha\beta}^{(1)}t + K_{\alpha\beta}^{(2)}t^2 + K_{\alpha\beta}^{(3)}t^3 + \dots, \quad (3.5)$$

and using the first variational equation (2.9), the coefficients $K_{\alpha\beta}^{(n)}$ are given by the general recursive formal expression

$$\gamma_{\alpha\beta}^{(n)} = \gamma_{\alpha\gamma}^{(0)}K_{\beta}^{(n)\gamma} + \sum_{\mu+\nu=n} \gamma_{\alpha\gamma}^{(\mu)}K_{\beta}^{(\nu)\gamma}. \quad (3.6)$$

In terms of the data a, b, c, d, e , we have explicitly,

$$K_{\alpha\beta} = \partial_t \gamma_{\alpha\beta} = b_{\alpha\beta} + 2c_{\alpha\beta}t + 3d_{\alpha\beta}t^2 + 4e_{\alpha\beta}t^3, \quad (3.7)$$

and for the mixed components we obtain,

$$\begin{aligned} K_{\beta}^{\alpha} = & \gamma^{\alpha\gamma}K_{\gamma\beta} = b_{\beta}^{\alpha} + (2c_{\beta}^{\alpha} - b^{\alpha\gamma}b_{\gamma\beta})t + (3d_{\beta}^{\alpha} - 2b^{\alpha\gamma}c_{\gamma\beta} + b^{\alpha\delta}b_{\delta}^{\gamma}b_{\gamma\beta} - c^{\alpha\gamma}b_{\gamma\beta})t^2 \\ & + (4e_{\beta}^{\alpha} - 3b^{\alpha\gamma}d_{\gamma\beta} + 2b^{\alpha\delta}b_{\delta}^{\gamma}c_{\gamma\beta} - 2c^{\alpha\gamma}c_{\gamma\beta} - d^{\alpha\gamma}b_{\gamma\beta} + b^{\alpha\delta}c_{\delta}^{\gamma}b_{\gamma\beta} - b^{\alpha\delta}b_{\delta}^{\epsilon}b_{\epsilon}^{\gamma}b_{\gamma\beta} \\ & + c^{\alpha\delta}b_{\delta}^{\gamma}b_{\gamma\beta})t^3. \end{aligned} \quad (3.8)$$

Further, setting $\gamma = |\gamma_{\alpha\beta}| = -g$, the mean curvature,

$$K = K_{\alpha}^{\alpha} = \gamma^{\alpha\beta}\partial_t \gamma_{\alpha\beta} = \partial_t \ln(\gamma), \quad (3.9)$$

is given by the form

$$\begin{aligned} K &= b + (2c - b_\beta^\alpha b_\alpha^\beta)t + (3d - 3b_\beta^\alpha c_\alpha^\beta + b^{\alpha\beta} b_\beta^\gamma b_{\gamma\alpha})t^2 \\ &+ (4e - 4b_\beta^\alpha d_\alpha^\beta + 4b^{\alpha\beta} c_\beta^\gamma b_{\gamma\alpha} - b^{\alpha\beta} b_\beta^\gamma b_\gamma^\delta b_{\delta\alpha} - 2c_\beta^\alpha c_\alpha^\beta)t^3. \end{aligned} \quad (3.10)$$

For completeness, we also give the expressions of the coefficients $K^{(n), \alpha\beta}$ of the fully contravariant symbols,

$$\begin{aligned} K^{\alpha\beta} &= b^{\alpha\beta} - 2(b^{\alpha\gamma} b_\gamma^\beta - c^{\alpha\beta})t - 3(-d^{\alpha\beta} + b^{\alpha\gamma} c_\gamma^\beta - b^{\alpha\gamma} b_\gamma^\delta b_\delta^\beta + c^{\alpha\gamma} b_\gamma^\beta)t^2 \\ &- 4(-e^{\alpha\beta} + b^{\alpha\gamma} d_\gamma^\beta - b^{\alpha\gamma} b_\gamma^\delta c_\delta^\beta + c^{\alpha\gamma} c_\gamma^\beta + d^{\alpha\gamma} b_\gamma^\beta - b^{\alpha\gamma} c_\gamma^\delta b_\delta^\beta \\ &+ b^{\alpha\gamma} b_\gamma^\delta b_\delta^\epsilon b_\epsilon^\beta - c^{\alpha\gamma} b_\gamma^\delta b_\delta^\beta)t^3. \end{aligned} \quad (3.11)$$

Using these forms, we can find the various components of the acceleration and jerk tensors to the required order. We have that the perturbation of the acceleration tensor in terms of the prescribed data is given by the form:

$$D_{\alpha\beta} = \partial_t K_{\alpha\beta} = 2c_{\alpha\beta} + 6d_{\alpha\beta}t + 12e_{\alpha\beta}t^2 + \dots \quad (3.12)$$

Further we find,

$$D_\beta^\alpha = \gamma^{\alpha\gamma} D_{\gamma\beta} = 2c_\beta^\alpha + 2(3d_\beta^\alpha - b_\gamma^\alpha c_\beta^\gamma)t + 2(6e_\beta^\alpha - 3b_\gamma^\alpha d_\beta^\gamma + b_\delta^\alpha b_\gamma^\delta c_\beta^\gamma - c_\gamma^\alpha c_\beta^\gamma)t^2, \quad (3.13)$$

and

$$D = 2c + 2(3d - b_\beta^\alpha c_\alpha^\beta)t + 2(6e - 3b_\beta^\alpha d_\alpha^\beta + b_\beta^\gamma b_\alpha^\beta c_\gamma^\alpha - c_\beta^\alpha c_\alpha^\beta)t^2, \quad (3.14)$$

where the trace is given by

$$D = D_\alpha^\alpha = \gamma^{\alpha\beta} \partial_t K_{\alpha\beta}.$$

For the fully contravariant components, we find,

$$\begin{aligned} D^{\alpha\beta} &= 2c^{\alpha\beta} + 2(3d^{\alpha\beta} - b^{\alpha\gamma} c_\gamma^\beta - b^{\delta\beta} c_\delta^\alpha)t \\ &+ 2(6e^{\alpha\beta} - 3b^{\alpha\gamma} d_\gamma^\beta - 3b^{\delta\beta} d_\delta^\alpha - 2c^{\alpha\gamma} c_\gamma^\beta + b^{\alpha\delta} b_\delta^\gamma c_\gamma^\beta + b^{\delta\beta} b_\gamma^\alpha c_\delta^\gamma + b^{\delta\gamma} b_\gamma^\beta c_\delta^\alpha)t^2. \end{aligned} \quad (3.15)$$

Lastly, the jerk perturbation series is found to be

$$W_{\alpha\beta} = \partial_t D_{\alpha\beta} = 6d_{\alpha\beta} + 24e_{\alpha\beta}t + \dots, \quad (3.16)$$

so that

$$W = 6d + 6(4e - b_\beta^\alpha d_\alpha^\beta)t + \dots \quad (3.17)$$

where

$$W = W_\alpha^\alpha = \gamma^{\alpha\beta} \partial_t D_{\alpha\beta}.$$

The components of the Ricci curvature are more complicated when expressed in terms of the asymptotic data a, b, c, d, e and we give them in the Appendix B. In terms of those and the various expressions obtained above, the hamiltonian constraint (2.16) becomes

$$\begin{aligned} \mathcal{C}_0 &= \frac{1}{2}P - \frac{1}{8}b_\alpha^\beta b_\beta^\alpha + \frac{1}{8}b^2 \\ &+ \epsilon \left[\frac{1}{4}P b_\alpha^\beta b_\beta^\alpha + 2c^2 - c b_\alpha^\beta b_\beta^\alpha + \frac{3}{32}(b_\alpha^\beta b_\beta^\alpha)^2 + \frac{1}{16}b^2 b_\alpha^\beta b_\beta^\alpha - \frac{1}{2}P^2 - \frac{1}{4}b^2 P - \frac{1}{32}b^4 \right. \\ &+ 2\alpha^{\beta\alpha} \nabla_\beta \nabla_\alpha P - 2\alpha^{\beta\alpha} \nabla_\beta \nabla_\alpha (-2c + \frac{3}{4}b_\gamma^\delta b_\delta^\gamma - \frac{1}{4}b^2) \Big] \\ &+ \left\{ (-\frac{1}{2}bc - \frac{1}{2}b_\alpha^\beta c_\beta^\alpha + \frac{1}{4}b_\alpha^\beta b_\gamma^\alpha b_\beta^\gamma) + \epsilon [2R^{(0)}(-3d + 2b_\alpha^\beta c_\beta^\alpha - \frac{1}{2}b_\alpha^\beta b_\gamma^\alpha b_\beta^\gamma) \right. \\ &+ 2R^{(1)}(-c + \frac{1}{4}b_\alpha^\beta b_\beta^\alpha) - R^{(0)}R^{(1)} - 2a^{\alpha\beta} \nabla_\alpha \nabla_\beta R^{(1)} + 2b_\beta^\alpha \nabla_\alpha \nabla_\beta R^{(0)}] \Big\} t + \dots = 0. \end{aligned} \quad (3.18)$$

From (2.17), we calculate the momentum constraint in the form

$$\begin{aligned} \mathcal{C}_\alpha &= \frac{1}{2}(\nabla_\beta b_\alpha^\beta - \nabla_\alpha b) + \epsilon \left[R^{(0)}(\nabla_\beta b_\alpha^\beta - \nabla_\alpha b) + 2\partial_t \nabla_\alpha P - 2\nabla_\alpha R^{(1)} + b_\alpha^\beta \nabla_\beta R^{(0)} \right] \\ &+ \left\{ [(\nabla_\beta c_\alpha^\beta - \nabla_\alpha c) - \frac{1}{2}\nabla_\beta (b_\gamma^\delta b_\delta^\gamma) + \frac{1}{2}\nabla_\alpha (b_\gamma^\beta b_\beta^\gamma)] \right. \\ &+ \epsilon \left[2R^{(0)} \left((\nabla_\beta c_\alpha^\beta - \nabla_\alpha c) - \frac{1}{2}\nabla_\beta (b_\gamma^\delta b_\delta^\gamma) + \frac{1}{2}\nabla_\alpha (b_\gamma^\beta b_\beta^\gamma) \right) + R^{(1)}(\nabla_\beta b_\alpha^\beta - \nabla_\alpha b) \right. \\ &\left. \left. - 4\nabla_\alpha R^{(2)} + b_\alpha^\beta \nabla_\beta R^{(1)} + (2c_\alpha^\beta - b_\gamma^\beta b_\alpha^\gamma) \nabla_\beta R^{(0)} \right] \right\} t + \dots = 0. \end{aligned} \quad (3.19)$$

Finally, the snap equation (2.18) gives:

$$\begin{aligned} &\frac{1}{2}P + 2c - \frac{5}{8}b_\alpha^\beta b_\beta^\alpha + \frac{1}{8}b^2 \\ &+ \epsilon \left\{ 2[-P + (-2c + \frac{3}{4}b_\gamma^\delta b_\delta^\gamma - \frac{1}{4}b^2)][-P + (-c + \frac{1}{2}b_\alpha^\beta b_\beta^\alpha - \frac{1}{4}b^2)] - \frac{3}{2}[-P + (-2c + \frac{3}{4}b_\gamma^\delta b_\delta^\gamma - \frac{1}{4}b^2)]^2 \right. \\ &+ 6[-\partial_t^2 P - 3bd - 2c^2 - 24e + 21b_\delta^\gamma d_\gamma^\delta + 10c_\delta^\gamma c_\gamma^\delta - 19b_\delta^\epsilon b_\gamma^\delta c_\epsilon^\gamma + \frac{9}{2}b_\gamma^\delta b_\epsilon^\gamma b_\zeta^\epsilon b_\delta^\zeta + 3b_\delta^\gamma c_\gamma^\delta b \\ &+ 2b_\delta^\gamma b_\gamma^\delta c - b_\delta^\gamma b_\epsilon^\delta b_\gamma^\epsilon b - \frac{1}{2}(b_\delta^\gamma b_\gamma^\delta)^2] + 4a^{\alpha\gamma}[\nabla_\gamma \nabla_\alpha P - \nabla_\gamma \nabla_\alpha (-2c + \frac{3}{4}b_\epsilon^\delta b_\delta^\epsilon - \frac{1}{4}b^2)] \Big\} + \dots = 0 \end{aligned} \quad (3.20)$$

We realize that because of the presence of the ϵ parameter in the theory, the resulting perturbation series become considerably more complicated in higher orders, due to terms of the order $O(\epsilon t)$. To

simplify our further work, we find it convenient to use the following notation:

$$\begin{aligned}
\mathcal{C}_0 &= \lambda_0^0 + (\lambda_\epsilon)_0^0 \epsilon + (\lambda_t)_0^0 t + (\lambda_{\epsilon t})_0^0 \epsilon t + \dots \\
\mathcal{C}_\alpha &= \lambda_\alpha^0 + (\lambda_\epsilon)_\alpha^0 \epsilon + (\lambda_t)_\alpha^0 t + (\lambda_{\epsilon t})_\alpha^0 \epsilon t + \dots \\
L_\alpha^\beta &= \lambda_\alpha^\beta + (\lambda_\epsilon)_\alpha^\beta \epsilon + \dots \\
\text{snap terms} &= \lambda + (\lambda_\epsilon) \epsilon + \dots
\end{aligned}$$

where the indices of the λ -coefficients indicate the powers of t, ϵ that appear in the series. For instance, the $O(t)$ term in the momentum constraint (3.19) is denoted by

$$(\lambda_t)_\alpha^0 = (\nabla_\beta c_\alpha^\beta - \nabla_\alpha c) - \frac{1}{2} \nabla_\beta (b_\gamma^\beta b_\alpha^\gamma) + \frac{1}{2} \nabla_\alpha (b_\gamma^\beta b_\beta^\gamma).$$

We are now ready to decide how much the imposition of the field equations (2.9), (2.10), (2.11) and (2.18) together with the constraints (2.16) and (2.17), restricts the number of free functions in the data (3.2) in the perturbation series (3.1). Using the field equations and calculating the various relations that appear at each order, we are led to the following relations between the data a, b, c, d, e : From λ_0^0, λ , we get one relation, namely,

$$c = \frac{1}{4} b_\alpha^\beta b_\beta^\alpha, \quad (3.21)$$

from λ_α^0 we obtain three more relations,

$$\nabla_\beta b_\alpha^\beta = \nabla_\alpha b, \quad (3.22)$$

whereas from $\lambda, \lambda_\alpha^\beta$ and Eq. (3.21) we get the following six relations,

$$-P_\alpha^\beta - \frac{1}{4} b_\alpha^\beta b + \frac{1}{2} b_\gamma^\beta b_\alpha^\gamma - c_\alpha^\beta = 0. \quad (3.23)$$

From λ and Eq. (3.21), the $(\lambda_\epsilon)_\alpha^\beta$ term gives one more relation

$$4c^2 + 6bd + 48e - 42b_\alpha^\beta d_\beta^\alpha - 20c_\alpha^\beta c_\beta^\alpha + 38b_\alpha^\beta b_\gamma^\alpha c_\beta^\gamma - 9b_\gamma^\beta b_\alpha^\gamma b_\delta^\alpha b_\beta^\delta - 6b_\alpha^\beta c_\beta^\alpha b + 2b_\beta^\gamma b_\alpha^\beta b_\gamma^\alpha b = 0, \quad (3.24)$$

and from Eq. (3.21), the $(\lambda_t)_0^0$ term gives the relation

$$2bc + 2b_\alpha^\beta c_\beta^\alpha - b_\alpha^\beta b_\gamma^\alpha b_\beta^\gamma = 0. \quad (3.25)$$

From Eq. (3.21) and λ , the $(\lambda_\epsilon)_\alpha^0$ term gives three more relations

$$\nabla_\alpha (bc - 6d + 5b_\gamma^\beta c_\beta^\gamma - \frac{3}{2} b_\delta^\beta b_\gamma^\delta b_\beta^\gamma) = 0. \quad (3.26)$$

Further, we note that the term $(\lambda_{\epsilon t})_0^0$ is identically zero due to the term λ and the equations (3.21), (3.26). The term $(\lambda_t)_\alpha^0$ is identically zero due to the term λ , the equations (3.21), (3.22),

(3.23) and the fact that $\nabla_\beta P_\alpha^\beta = \frac{1}{2}\nabla_\alpha P$. The term $(\lambda_{et})_\alpha^0$ is identically zero due to the equations (3.21), (3.22) and the term $(\lambda_t)_\alpha^0$. The terms $(\lambda_\epsilon)_0^0$ and λ_ϵ are identically zero due to the equations (3.21), (3.23).

Hence, in total we find that the imposition of the field equations leads to 15 relations between the 30 functions of the perturbation metric (3.1), that is we are left with 15 free functions. Taking into account the freedom we have in performing 4 diffeomorphism changes, we finally conclude that there are in total 11 free functions in the solution (3.1). This means that the regular solution (3.1) corresponds to a general solution of the problem. Put it differently, ‘regularity is a generic feature of the $R + \epsilon R^2$ theory in vacuum’.

We now ask: Out of the 30 different functions a, b, c, d, e , which eleven of those are we to choose to use as our initial data? We have shown in this Section that the vacuum higher order gravity equations (2.9), (2.10), (2.11) and (2.18) together with the constraints (2.16) and (2.17), admit a regular formal series expansion of the form (3.1) as a general solution requiring 11 smooth initial data. If we prescribe the thirty data

$$a_{\alpha\beta}, \quad b_{\alpha\beta}, \quad c_{\alpha\beta}, \quad d_{\alpha\beta}, \quad e_{\alpha\beta}, \quad (3.27)$$

initially, we still have the freedom to fix 19 of them. We choose to leave the six components of the metric $a_{\alpha\beta}$ free, and we choose the four symmetric space tensors $b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta}$ and $e_{\alpha\beta}$ to be traceless with respect to $a_{\alpha\beta}$. Then we proceed to count the number of free functions in several steps. In the first step, using the trace condition on $b_{\alpha\beta}$, $b = 0$, (3.22) gives

$$\nabla^\alpha b_\alpha^\beta = 0. \quad (3.28)$$

Hence only two of the six components of $b_{\alpha\beta}$ can be free. Secondly, if $c_{\alpha\beta}$ has zero trace, then Eq. (3.21) fixes one more component of $b_{\alpha\beta}$, thus we end up with 7 free functions (6 from $a_{\alpha\beta}$ and one more from $b_{\alpha\beta}$). Then, as step three, we use the 6 relations in (3.23) to completely fix the remaining 6 components of $b_{\alpha\beta}$ and $c_{\alpha\beta}$, namely, they fix the one remaining component of $b_{\alpha\beta}$ and the 5 ones of $c_{\alpha\beta}$ from step 2. Step four consists in taking into account the five remaining relations (3.24), (3.25) and (3.26): We find that only seven from the twelve remaining components of the tensors $d_{\alpha\beta}$ and $e_{\alpha\beta}$ can be independent. But since we took $d_{\alpha\beta}$ and $e_{\alpha\beta}$ to be traceless, as a last step in our function counting, we find that together they can have 5 free, independent components at most. Summing up the free functions we found up to now, we end up with

$$\underbrace{6}_{\text{from } a_{\alpha\beta}} + \underbrace{0}_{\text{from } b_{\alpha\beta}} + \underbrace{0}_{\text{from } c_{\alpha\beta}} + \underbrace{5}_{\text{from } d_{\alpha\beta} \text{ and } e_{\alpha\beta}} = 11 \quad (3.29)$$

suitable free data as required for the solution to be a general one. We thus arrive at the following result which summarizes what we have shown in this Section, and generalizes a theorem of Rendall [18] for higher order gravity theories that derive from the lagrangian $R + \epsilon R^2$.

Theorem 3.1 *Let $a_{\alpha\beta}$ be a smooth Riemannian metric, $b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta}$ and $e_{\alpha\beta}$ be symmetric smooth tensor fields which are traceless with respect to the metric $a_{\alpha\beta}$, i.e., they satisfy $b = c = d = e = 0$. Then there exists a formal power series expansion solution of the vacuum higher order gravity equations of the form (3.1) such that:*

1. *It is unique*
2. *The coefficients $\gamma^{(n)\alpha\beta}$ are all smooth*
3. *It holds that $\gamma_{\alpha\beta}^{(0)} = a_{\alpha\beta}$ and $\gamma_{\alpha\beta}^{(1)} = b_{\alpha\beta}$, $\gamma_{\alpha\beta}^{(2)} = c_{\alpha\beta}$, $\gamma_{\alpha\beta}^{(3)} = d_{\alpha\beta}$ and $\gamma_{\alpha\beta}^{(4)} = e_{\alpha\beta}$.*

In the course of the proof of this result, uniqueness followed because all coefficients were found recursively, while smoothness follows because in no step of the proof did we find it necessary to lower the C^∞ assumption. We also note that $b_{\alpha\beta}$ and $c_{\alpha\beta}$ are necessarily transverse with respect to $a_{\alpha\beta}$.

4 Discussion

In this paper, we have treated the problem of the existence of generic perturbations of the regular state in higher order gravity in vacuum that derives from the lagrangian $R + \epsilon R^2$. We have shown that there is a regular state of the theory in the form of a formal series expansion having the same number of free functions as those required for a general solution of the theory. This means that there exists an open set in the space of initial data of the theory that leads to a regular solution having the correct number of free functions to qualify as a general solution.

To achieve this, we have shown that there exists a first order formulation of the theory with the Cauchy-Kovalevski property. This formulation of the quadratic theory $R + \epsilon R^2$ evolves an initial data set $(\mathcal{M}, \gamma_{\alpha\beta}, K_{\alpha\beta}, D_{\alpha\beta}, W)$ through a set of the four evolution equations (2.9), (2.10), (2.11) and (2.18) and the two constraints (2.16) and (2.17), and builds the time development (\mathcal{V}, g) . What we have proved is that if we start with an initial data set in which the metric has the asymptotic form (3.1) and evolve, then we can build an asymptotic development in the form of a formal series expansion which satisfies the evolution and constraint equations and has the same number of free functions as those of a general solution of the theory. In other words, we have shown that regularity is a generic feature of the $R + \epsilon R^2$ theory under the assumption of analyticity.

The results of this paper provide the necessary background for various further investigations that we carry out currently. For example, elsewhere we plan to examine the generic perturbation problem of the well known radiation solution of the quadratic theory and extend the results of this paper to any perfect fluid spacetime with equation of state $p = w\rho$ in higher order gravity. It is also interesting to further compare these radiation perturbations with the situation in vacuum in the context of higher order gravity. We also wish to extend our present results to the case of arbitrary lapse and shift, and to consider the problem of the present paper in the conformal frame. We know that in the case of a positive cosmological constant and a general perfect fluid source, similar results to those of this paper hold, cf. [18]. However, the scalar field case, especially with the potential of the Einstein frame representation of an $f(R)$ theory, is to our knowledge an open problem.

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Appendix A: Proof of the Cauchy-Kovalevski property

In this Appendix we present details of the proof that the four evolution equations (2.9), (2.10), (2.11) and (2.18) constitute a Cauchy-Kovalevski type system. Indeed, the only ‘dangerous’ terms present in these equations are the three terms

$$\partial_t P, \quad \partial_t(\partial_t P), \quad \nabla_\alpha \nabla_\beta (-P - D + \frac{3}{4} K^{\gamma\delta} K_{\gamma\delta} - \frac{1}{4} K^2), \quad (4.1)$$

present in the constraints and in the snap equation (2.18).

The spatial connection coefficients are given by the obvious formula

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2} \gamma^{\mu\epsilon} (\partial_\beta \gamma_{\alpha\epsilon} + \partial_\alpha \gamma_{\beta\epsilon} - \partial_\epsilon \gamma_{\alpha\beta}). \quad (4.2)$$

We then have

$$\partial_t \Gamma_{\alpha\beta}^\mu = -\frac{1}{2} K^{\mu\epsilon} (\partial_\beta \gamma_{\alpha\epsilon} + \partial_\alpha \gamma_{\beta\epsilon} - \partial_\epsilon \gamma_{\alpha\beta}) + \frac{1}{2} \gamma^{\mu\epsilon} (\partial_\beta K_{\alpha\epsilon} + \partial_\alpha K_{\beta\epsilon} - \partial_\epsilon K_{\alpha\beta}), \quad (4.3)$$

and

$$\begin{aligned} \partial_t^2 \Gamma_{\alpha\beta}^\mu &= -K^{\mu\epsilon} (\partial_\beta K_{\alpha\epsilon} + \partial_\alpha K_{\beta\epsilon} - \partial_\epsilon K_{\alpha\beta}) \\ &\quad - \frac{1}{2} (D^{\mu\epsilon} - 2K^{\mu\eta} K_\eta^\epsilon) (\partial_\beta \gamma_{\alpha\epsilon} + \partial_\alpha \gamma_{\beta\epsilon} - \partial_\epsilon \gamma_{\alpha\beta}) \\ &\quad + \frac{1}{2} \gamma^{\mu\epsilon} (\partial_\beta D_{\alpha\epsilon} + \partial_\alpha D_{\beta\epsilon} - \partial_\epsilon D_{\alpha\beta}). \end{aligned} \quad (4.4)$$

We also set

$$\Gamma_{\eta\alpha\beta} = \frac{1}{2}(\partial_\beta\gamma_{\alpha\eta} + \partial_\alpha\gamma_{\beta\eta} - \partial_\eta\gamma_{\alpha\beta}), \quad (4.5)$$

so that $\Gamma_{\alpha\beta}^\mu = \gamma^{\mu\eta}\Gamma_{\eta\alpha\beta}$, and further we set

$$Z_{\eta\alpha\beta} = \frac{1}{2}(\partial_\beta K_{\alpha\eta} + \partial_\alpha K_{\beta\eta} - \partial_\eta K_{\alpha\beta}), \quad (4.6)$$

and

$$H_{\eta\alpha\beta} = \frac{1}{2}(\partial_\beta D_{\alpha\eta} + \partial_\alpha D_{\beta\eta} - \partial_\eta D_{\alpha\beta}). \quad (4.7)$$

It then follows that the time derivatives of these ‘fully covariant symbols’ satisfy

$$\partial_t \Gamma_{\eta\alpha\beta} = Z_{\eta\alpha\beta}, \quad (4.8)$$

$$\partial_t Z_{\eta\alpha\beta} = H_{\eta\alpha\beta}, \quad (4.9)$$

$$\partial_t^2 \Gamma_{\eta\alpha\beta} = H_{\eta\alpha\beta}. \quad (4.10)$$

Hence we conclude that the first time derivatives of the spatial connection coefficients depend only of $(\gamma_{\alpha\beta}, K_{\alpha\beta})$ and their first *spatial* derivatives, and the second time derivatives of the spatial connection coefficients depend only of $(\gamma_{\alpha\beta}, K_{\alpha\beta}, D_{\alpha\beta})$ and their first *spatial* derivatives.

Now, the spatial Ricci tensor is given by

$$P_{\alpha\beta} = \partial_\mu \Gamma_{\alpha\beta}^\mu - \partial_\beta \Gamma_{\alpha\mu}^\mu + \Gamma_{\alpha\beta}^\mu \Gamma_{\mu\epsilon}^\epsilon - \Gamma_{\alpha\epsilon}^\mu \Gamma_{\beta\mu}^\epsilon, \quad (4.11)$$

and so its time derivative is calculated to be

$$\partial_t P_{\alpha\beta} = \partial_\mu (\partial_t \Gamma_{\alpha\beta}^\mu) - \partial_\beta (\partial_t \Gamma_{\alpha\mu}^\mu) + \partial_t \Gamma_{\alpha\beta}^\mu \Gamma_{\mu\epsilon}^\epsilon + \Gamma_{\alpha\beta}^\mu \partial_t \Gamma_{\mu\epsilon}^\epsilon - \partial_t \Gamma_{\alpha\epsilon}^\mu \Gamma_{\beta\mu}^\epsilon - \Gamma_{\alpha\epsilon}^\mu \partial_t \Gamma_{\beta\mu}^\epsilon. \quad (4.12)$$

Then, for the spatial scalar curvature,

$$P = \gamma^{\alpha\beta} P_{\alpha\beta}, \quad (4.13)$$

we find that

$$\partial_t P = -K^{\alpha\beta} P_{\alpha\beta} + \gamma^{\alpha\beta} \partial_t P_{\alpha\beta}, \quad (4.14)$$

and therefore the first of the dangerous terms finally reads:

$$\begin{aligned} \partial_t P &= K^{\alpha\beta} (-\partial_\mu \gamma^{\mu\epsilon} \Gamma_{\epsilon\alpha\beta} - \gamma^{\mu\epsilon} \partial_\mu \Gamma_{\epsilon\alpha\beta} + \partial_\beta \gamma^{\mu\epsilon} \Gamma_{\epsilon\alpha\mu} + \gamma^{\mu\epsilon} \partial_\beta \Gamma_{\epsilon\alpha\mu} - \gamma^{\mu\epsilon} \gamma^{\zeta\eta} \Gamma_{\epsilon\alpha\beta} \Gamma_{\eta\mu\zeta} + \gamma^{\mu\eta} \gamma^{\epsilon\zeta} \Gamma_{\eta\alpha\epsilon} \Gamma_{\zeta\beta\mu}) \\ &+ \gamma^{\alpha\beta} [-\partial_\mu K^{\mu\epsilon} \Gamma_{\epsilon\alpha\beta} - K^{\mu\epsilon} \partial_\mu \Gamma_{\epsilon\alpha\beta} + \partial_\mu \gamma^{\mu\epsilon} Z_{\epsilon\alpha\beta} + \gamma^{\mu\epsilon} \partial_\mu Z_{\epsilon\alpha\beta} - \partial_\beta K^{\mu\epsilon} \Gamma_{\epsilon\alpha\mu} - K^{\mu\epsilon} \partial_\beta \Gamma_{\epsilon\alpha\mu} \\ &+ \partial_\beta \gamma^{\mu\epsilon} Z_{\epsilon\alpha\mu} + \gamma^{\mu\epsilon} \partial_\beta Z_{\epsilon\alpha\mu} \\ &+ \gamma^{\zeta\eta} \Gamma_{\eta\mu\zeta} (-K^{\mu\epsilon} \Gamma_{\epsilon\alpha\beta} + \gamma^{\mu\epsilon} Z_{\epsilon\alpha\beta}) + \gamma^{\mu\zeta} \Gamma_{\zeta\alpha\beta} (-K^{\epsilon\eta} \Gamma_{\eta\mu\epsilon} + \gamma^{\epsilon\eta} Z_{\eta\mu\epsilon}) \\ &- \gamma^{\epsilon\zeta} \Gamma_{\zeta\beta\mu} (-K^{\mu\eta} \Gamma_{\eta\alpha\epsilon} + \gamma^{\mu\eta} Z_{\eta\alpha\epsilon}) - \gamma^{\mu\zeta} \Gamma_{\zeta\alpha\epsilon} (-K^{\epsilon\eta} \Gamma_{\eta\beta\mu} + \gamma^{\epsilon\eta} Z_{\eta\beta\mu})]. \end{aligned} \quad (4.15)$$

This means that the first dangerous term is ‘purely spatial’. This result also implies that the third dangerous term is also purely spatial, for it is calculated to be of the form

$$\begin{aligned}
\nabla_\alpha \nabla_\beta (-P - D + \frac{3}{4} K^{\gamma\delta} K_{\gamma\delta} - \frac{1}{4} K^2) &= \partial_\alpha [\partial_\beta (-P - D + \frac{3}{4} K^{\gamma\delta} K_{\gamma\delta} - \frac{1}{4} K^2)] \\
&- \Gamma_{\alpha\beta}^\mu \partial_\mu (-P - D + \frac{3}{4} K^{\gamma\delta} K_{\gamma\delta} - \frac{1}{4} K^2) \\
&- \frac{1}{2} K_{\alpha\beta} (-\partial_t P - W + \frac{5}{2} K^{\alpha\beta} D_{\alpha\beta} \\
&- \frac{3}{2} K^{\alpha\gamma} K_\gamma^\beta K_{\alpha\beta} - \frac{1}{2} K D + \frac{1}{2} K K^{\alpha\beta} K_{\alpha\beta}), \quad (4.16)
\end{aligned}$$

that is trouble could only have arisen from the first dangerous term, which however as we showed above is purely spatial.

Lastly, since

$$\partial_t (\partial_t P) = -(D^{\alpha\beta} - 2K^{\alpha\gamma} K_\gamma^\beta) P_{\alpha\beta} - 2K^{\alpha\beta} \partial_t P_{\alpha\beta} + \gamma^{\alpha\beta} \partial_t (\partial_t P_{\alpha\beta}), \quad (4.17)$$

and

$$\begin{aligned}
\partial_t^2 P_{\alpha\beta} &= \partial_\mu (\partial_t^2 \Gamma_{\alpha\beta}^\mu) - \partial_\beta (\partial_t^2 \Gamma_{\alpha\mu}^\mu) + \partial_t^2 \Gamma_{\alpha\beta}^\mu \Gamma_{\mu\epsilon}^\epsilon + \Gamma_{\alpha\beta}^\mu \partial_t^2 \Gamma_{\mu\epsilon}^\epsilon - \partial_t^2 \Gamma_{\alpha\epsilon}^\mu \Gamma_{\beta\mu}^\epsilon \\
&- \partial_t^2 \Gamma_{\beta\mu}^\epsilon \Gamma_{\alpha\epsilon}^\mu + 2\partial_t \Gamma_{\alpha\beta}^\mu \partial_t \Gamma_{\mu\epsilon}^\epsilon - 2\partial_t \Gamma_{\alpha\epsilon}^\mu \partial_t \Gamma_{\beta\mu}^\epsilon, \quad (4.18)
\end{aligned}$$

we find that the second dangerous term depends on $(\gamma_{\alpha\beta}, K_{\alpha\beta}, D_{\alpha\beta})$ and its first and second spatial

derivatives, namely, it has the form:

$$\begin{aligned}
\partial_t(\partial_t P) = & (D^{\alpha\beta} - 2K^{\alpha\gamma}K_\gamma^\beta)(-\partial_\mu\gamma^{\mu\epsilon}\Gamma_{\epsilon\alpha\beta} - \gamma^{\mu\epsilon}\partial_\mu\Gamma_{\epsilon\alpha\beta} + \partial_\beta\gamma^{\mu\epsilon}\Gamma_{\epsilon\alpha\mu} + \gamma^{\mu\epsilon}\partial_\beta\Gamma_{\epsilon\alpha\mu} \\
& - \gamma^{\mu\epsilon}\gamma^{\zeta\eta}\Gamma_{\epsilon\alpha\beta}\Gamma_{\eta\mu\zeta} + \gamma^{\mu\eta}\gamma^{\epsilon\zeta}\Gamma_{\eta\alpha\epsilon}\Gamma_{\zeta\beta\mu}) \\
& - 2K^{\alpha\beta}[-\partial_\mu K^{\mu\epsilon}\Gamma_{\epsilon\alpha\beta} - K^{\mu\epsilon}\partial_\mu\Gamma_{\epsilon\alpha\beta} + \partial_\mu\gamma^{\mu\epsilon}Z_{\epsilon\alpha\beta} + \gamma^{\mu\epsilon}\partial_\mu Z_{\epsilon\alpha\beta} - \partial_\beta K^{\mu\epsilon}\Gamma_{\epsilon\alpha\mu} - K^{\mu\epsilon}\partial_\beta\Gamma_{\epsilon\alpha\mu} \\
& + \partial_\beta\gamma^{\mu\epsilon}Z_{\epsilon\alpha\mu} + \gamma^{\mu\epsilon}\partial_\beta Z_{\epsilon\alpha\mu} \\
& + \gamma^{\zeta\eta}\Gamma_{\eta\mu\zeta}(-K^{\mu\epsilon}\Gamma_{\epsilon\alpha\beta} + \gamma^{\mu\epsilon}Z_{\epsilon\alpha\beta}) + \gamma^{\mu\zeta}\Gamma_{\zeta\alpha\beta}(-K^{\epsilon\eta}\Gamma_{\eta\mu\epsilon} + \gamma^{\epsilon\eta}Z_{\eta\mu\epsilon}) \\
& - \gamma^{\epsilon\zeta}\Gamma_{\zeta\beta\mu}(-K^{\mu\eta}\Gamma_{\eta\alpha\epsilon} + \gamma^{\mu\eta}Z_{\eta\alpha\epsilon}) - \gamma^{\mu\zeta}\Gamma_{\zeta\alpha\epsilon}(-K^{\epsilon\eta}\Gamma_{\eta\beta\mu} + \gamma^{\epsilon\eta}Z_{\eta\beta\mu})] \\
& + \gamma^{\alpha\beta}\{-\partial_\mu(D^{\mu\epsilon} - 2K^{\mu\gamma}K_\gamma^\epsilon)\Gamma_{\epsilon\alpha\beta} - (D^{\mu\epsilon} - 2K^{\mu\gamma}K_\gamma^\epsilon)\partial_\mu\Gamma_{\epsilon\alpha\beta} \\
& - 2\partial_\mu K^{\mu\epsilon}Z_{\epsilon\alpha\beta} - 2K^{\mu\epsilon}\partial_\mu Z_{\epsilon\alpha\beta} + \partial_\mu\gamma^{\mu\epsilon}H_{\epsilon\alpha\beta} + \gamma^{\mu\epsilon}\partial_\mu H_{\epsilon\alpha\beta} \\
& + \partial_\beta(D^{\mu\epsilon} - 2K^{\mu\gamma}K_\gamma^\epsilon)\Gamma_{\epsilon\alpha\mu} + (D^{\mu\epsilon} - 2K^{\mu\gamma}K_\gamma^\epsilon)\partial_\beta\Gamma_{\epsilon\alpha\mu} \\
& + 2\partial_\beta K^{\mu\epsilon}Z_{\epsilon\alpha\mu} + 2K^{\mu\epsilon}\partial_\beta Z_{\epsilon\alpha\mu} - \partial_\beta\gamma^{\mu\epsilon}H_{\epsilon\alpha\mu} - \gamma^{\mu\epsilon}\partial_\beta H_{\epsilon\alpha\mu} \\
& + \gamma^{\zeta\eta}[-(D^{\mu\epsilon} - 2K^{\mu\gamma}K_\gamma^\epsilon)\Gamma_{\epsilon\alpha\beta} - 2K^{\mu\epsilon}Z_{\epsilon\alpha\beta} + \gamma^{\mu\epsilon}H_{\epsilon\alpha\beta}]\Gamma_{\eta\mu\zeta} \\
& + \gamma^{\mu\zeta}[-(D^{\epsilon\eta} - 2K^{\epsilon\gamma}K_\gamma^\eta)\Gamma_{\eta\mu\epsilon} - 2K^{\epsilon\eta}Z_{\eta\mu\epsilon} + \gamma^{\epsilon\eta}H_{\eta\mu\epsilon}]\Gamma_{\zeta\alpha\beta} \\
& - \gamma^{\epsilon\zeta}[-(D^{\mu\eta} - 2K^{\mu\gamma}K_\gamma^\eta)\Gamma_{\eta\alpha\epsilon} - 2K^{\mu\eta}Z_{\eta\alpha\epsilon} + \gamma^{\mu\eta}H_{\eta\alpha\epsilon}]\Gamma_{\zeta\beta\mu} \\
& - \gamma^{\mu\zeta}[-(D^{\epsilon\eta} - 2K^{\epsilon\gamma}K_\gamma^\eta)\Gamma_{\eta\beta\mu} - 2K^{\epsilon\eta}Z_{\eta\beta\mu} + \gamma^{\epsilon\eta}H_{\eta\beta\mu}]\Gamma_{\zeta\alpha\epsilon} \\
& + 2(-K^{\mu\epsilon}\Gamma_{\epsilon\alpha\beta} + \gamma^{\mu\epsilon}Z_{\epsilon\alpha\beta})(-K^{\zeta\eta}\Gamma_{\eta\mu\zeta} + \gamma^{\zeta\eta}Z_{\eta\mu\zeta}) \\
& - 2(-K^{\mu\eta}\Gamma_{\eta\alpha\epsilon} + \gamma^{\mu\eta}Z_{\eta\alpha\epsilon})(-K^{\epsilon\zeta}\Gamma_{\zeta\beta\mu} + \gamma^{\epsilon\zeta}Z_{\zeta\beta\mu})\}.
\end{aligned} \tag{4.19}$$

This completes the proof that the evolution equations are a Cauchy-Kovalevski system.

Appendix B: Ricci curvarure in terms of the data a, b, c, d, e

We give here the various components of the Ricci curvature and the space-space components of the field equation (2.5) in terms of the data a, b, c, d, e . We have:

$$\begin{aligned}
R_0^0 = & -\frac{1}{2}D + \frac{1}{4}K_\alpha^\beta K_\beta^\alpha = \left(-c + \frac{1}{4}b_\alpha^\beta b_\beta^\alpha\right) + \left(-3d + 2b_\alpha^\beta c_\beta^\alpha - \frac{1}{2}b_\alpha^\beta b_\gamma^\alpha b_\beta^\gamma\right)t \\
& + \left(-6e + \frac{9}{2}b_\alpha^\beta d_\beta^\alpha - \frac{7}{2}b_\alpha^\beta b_\gamma^\alpha c_\beta^\gamma + \frac{3}{4}b_\gamma^\beta b_\alpha^\gamma b_\delta^\alpha b_\beta^\delta + 2c_\alpha^\beta c_\beta^\alpha\right)t^2 + \dots,
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
R_\alpha^0 &= \frac{1}{2}(\nabla_\beta K_\alpha^\beta - \nabla_\alpha K) = \frac{1}{2}(\nabla_\beta b_\alpha^\beta - \nabla_\alpha b) + \left(\nabla_\beta c_\alpha^\beta - \nabla_\alpha c - \frac{1}{2}\nabla_\beta(b_\gamma^\beta b_\alpha^\gamma) + \frac{1}{2}\nabla_\alpha(b_\gamma^\beta b_\beta^\gamma) \right) t \\
&+ \left[\frac{3}{2}(\nabla_\beta d_\alpha^\beta - \nabla_\alpha d) - \nabla_\beta(b_\gamma^\beta c_\alpha^\gamma) + \frac{3}{2}\nabla_\alpha(b_\gamma^\beta c_\beta^\gamma) - \frac{1}{2}\nabla_\beta(c_\gamma^\beta b_\alpha^\gamma) + \frac{1}{2}\nabla_\beta(b_\delta^\beta b_\gamma^\delta b_\alpha^\gamma) - \frac{1}{2}\nabla_\alpha(b_\beta^\delta b_\gamma^\beta b_\delta^\gamma) \right] t^2 \\
&+ \left[2(\nabla_\beta e_\alpha^\beta - \nabla_\alpha e) - \frac{3}{2}\nabla_\beta(b_\gamma^\beta d_\alpha^\gamma) + 2\nabla_\alpha(b_\gamma^\beta d_\beta^\gamma) - \frac{1}{2}\nabla_\beta(d_\gamma^\beta b_\alpha^\gamma) - \nabla_\beta(c_\gamma^\beta c_\alpha^\gamma) + \nabla_\alpha(c_\gamma^\beta c_\beta^\gamma) \right. \\
&+ \nabla_\beta(b_\delta^\beta b_\gamma^\delta c_\alpha^\gamma) + \frac{1}{2}\nabla_\beta(b_\delta^\beta c_\gamma^\delta b_\alpha^\gamma) + \frac{1}{2}\nabla_\beta(c_\delta^\beta b_\gamma^\delta b_\alpha^\gamma) - 2\nabla_\alpha(b_\beta^\delta c_\gamma^\beta b_\delta^\gamma) - \frac{1}{2}\nabla_\beta(b_\delta^\beta b_\epsilon^\delta b_\gamma^\epsilon b_\alpha^\gamma) + \frac{1}{2}\nabla_\alpha(b_\beta^\delta b_\gamma^\beta b_\epsilon^\delta b_\delta^\gamma) \left. \right] t^3 \\
&+ \dots, \tag{4.21}
\end{aligned}$$

$$\begin{aligned}
R_\alpha^\beta &= -P_\alpha^\beta - \frac{1}{4}KK_\alpha^\beta - \frac{1}{2}D_\alpha^\beta + \frac{1}{2}K_\gamma^\beta K_\alpha^\gamma = \left(-P_\alpha^\beta - c_\alpha^\beta + \frac{1}{2}b_\gamma^\beta b_\alpha^\gamma - \frac{1}{4}b_\alpha^\beta b \right) \\
&+ \left(-3d_\alpha^\beta + 2b_\gamma^\beta c_\alpha^\gamma - b_\delta^\beta b_\gamma^\delta b_\alpha^\gamma + c_\gamma^\beta b_\alpha^\gamma - \frac{1}{2}b_\alpha^\beta c + \frac{1}{4}b_\gamma^\delta b_\delta^\gamma b_\alpha^\beta - \frac{1}{2}c_\alpha^\beta b + \frac{1}{4}b_\gamma^\beta b_\alpha^\gamma b \right) t \\
&+ \left(-6e_\alpha^\beta + \frac{9}{2}b_\gamma^\beta d_\alpha^\gamma - 3b_\delta^\beta b_\gamma^\delta c_\alpha^\gamma + 3c_\gamma^\beta c_\alpha^\gamma + \frac{3}{2}d_\gamma^\beta b_\alpha^\gamma - \frac{3}{2}b_\delta^\beta c_\gamma^\delta b_\alpha^\gamma + \frac{3}{2}b_\delta^\beta b_\epsilon^\delta b_\gamma^\epsilon b_\alpha^\gamma - \frac{3}{2}c_\delta^\beta b_\gamma^\delta b_\alpha^\gamma \right. \\
&- \frac{3}{4}d_\alpha^\beta b + \frac{1}{2}b_\gamma^\beta c_\alpha^\gamma b - \frac{1}{4}b_\delta^\beta b_\gamma^\delta b_\alpha^\gamma b + \frac{1}{4}c_\gamma^\beta b_\alpha^\gamma b - c_\alpha^\beta c + \frac{1}{2}c_\alpha^\beta b_\gamma^\delta b_\delta^\gamma + \frac{1}{2}b_\gamma^\beta b_\alpha^\gamma c - \frac{1}{4}b_\gamma^\beta b_\alpha^\gamma b_\epsilon^\delta b_\delta^\epsilon \\
&\left. - \frac{3}{4}b_\alpha^\beta d + \frac{3}{4}b_\alpha^\beta b_\gamma^\delta c_\delta^\gamma - \frac{1}{4}b_\alpha^\beta b_\epsilon^\delta b_\gamma^\epsilon b_\delta^\gamma \right) t^2 + \dots, \tag{4.22}
\end{aligned}$$

where $P_{\alpha\beta}$ the Ricci tensor associated with $\gamma_{\alpha\beta}$. The scalar curvature becomes an expression of the form,

$$R = R^{(0)} + R^{(1)}t + R^{(2)}t^2 + \dots, \tag{4.23}$$

explicitly we have,

$$\begin{aligned}
R &= -P - \frac{1}{4}K^2 + \frac{3}{4}K_\alpha^\beta K_\beta^\alpha - D \\
&= \left(-P - 2c + \frac{3}{4}b_\alpha^\beta b_\beta^\alpha - \frac{1}{4}b^2 \right) + \left(-bc - 6d + 5b_\alpha^\beta c_\beta^\alpha - \frac{3}{2}b_\alpha^\beta b_\gamma^\alpha b_\beta^\gamma + \frac{1}{2}b_\alpha^\beta b_\beta^\alpha b \right) t \\
&+ \left(-\frac{3}{2}bd - c^2 - 12e + \frac{21}{2}b_\alpha^\beta d_\beta^\alpha + 5c_\alpha^\beta c_\beta^\alpha - \frac{19}{2}b_\alpha^\beta b_\gamma^\alpha c_\beta^\gamma + \frac{9}{4}b_\gamma^\beta b_\alpha^\gamma b_\delta^\alpha b_\beta^\delta + \frac{3}{2}b_\alpha^\beta c_\beta^\alpha b \right. \\
&\left. + b_\alpha^\beta b_\beta^\alpha c - \frac{1}{2}b_\beta^\gamma b_\alpha^\beta b_\gamma^\alpha b - \frac{1}{4}(b_\alpha^\beta b_\beta^\alpha)^2 \right) t^2 + \dots. \tag{4.24}
\end{aligned}$$

Using these forms, the space-space components of the field equations in a Cauchy adapted frame, (2.5), become:

$$\begin{aligned}
L_\alpha^\beta &= -P_\alpha^\beta + (-c_\alpha^\beta + \frac{1}{2}b_\gamma^\beta b_\alpha^\gamma - \frac{1}{4}b_\alpha^\beta b) - \frac{1}{2}[-P + (-2c + \frac{3}{4}b_\gamma^\delta b_\delta^\gamma - \frac{1}{4}b^2)]\delta_\alpha^\beta \\
&+ \epsilon \left\{ 2[-P + (-2c + \frac{3}{4}b_\gamma^\delta b_\delta^\gamma - \frac{1}{4}b^2)][-P_\alpha^\beta + (-c_\alpha^\beta + \frac{1}{2}b_\gamma^\beta b_\alpha^\gamma - \frac{1}{4}b_\alpha^\beta b)] \right. \\
&- \frac{1}{2}[-P + (-2c + \frac{3}{4}b_\gamma^\delta b_\delta^\gamma - \frac{1}{4}b^2)]^2 \delta_\alpha^\beta - 2a^{\beta\gamma}[\nabla_\gamma \nabla_\alpha P - (-2c + \frac{3}{4}b_\epsilon^\delta b_\delta^\epsilon - \frac{1}{4}b^2)_{;\alpha;\gamma}] \\
&+ 2[-\frac{\partial^2}{\partial t^2}P - 3bd - 2c^2 - 24e + 21b_\delta^\gamma d_\gamma^\delta + 10c_\delta^\gamma c_\gamma^\delta - 19b_\delta^\epsilon b_\gamma^\delta c_\epsilon^\gamma + \frac{9}{2}b_\gamma^\delta b_\epsilon^\gamma b_\zeta^\epsilon b_\delta^\zeta + 3b_\delta^\gamma c_\gamma^\delta b \\
&+ 2b_\delta^\gamma b_\gamma^\delta c - b_\delta^\gamma b_\epsilon^\delta b_\gamma^\epsilon b - \frac{1}{2}(b_\delta^\gamma b_\gamma^\delta)^2] \delta_\alpha^\beta + 2a^{\gamma\delta}[\nabla_\gamma \nabla_\delta P - \nabla_\gamma \nabla_\delta(-2c + \frac{3}{4}b_\epsilon^\zeta b_\zeta^\epsilon - \frac{1}{4}b^2)] \delta_\alpha^\beta \Big\} \\
&+ \dots = 0.
\end{aligned} \tag{4.25}$$

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